

HYPERELLIPTIC JACOBIANS AND MODULAR REPRESENTATIONS

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1. INTRODUCTION

In [17] the author proved that in characteristic 0 the jacobian $J(C) = J(C_f)$ of a hyperelliptic curve

$$C = C_f : y^2 = f(x)$$

has only trivial endomorphisms over an algebraic closure K_a of the ground field K if the Galois group $\text{Gal}(f)$ of the irreducible polynomial $f \in K[x]$ is “very big”. Namely, if $n = \deg(f) \geq 5$ and $\text{Gal}(f)$ is either the symmetric group \mathbf{S}_n or the alternating group \mathbf{A}_n then the ring $\text{End}(J(C_f))$ of K_a -endomorphisms of $J(C_f)$ coincides with \mathbf{Z} . The proof was based on an explicit description of the Galois module $J(C_f)_2$ of points of order 2 on $J(C_f)$. Namely, the action of the Galois group $\text{Gal}(K)$ factors through $\text{Gal}(f)$ and the $\text{Gal}(f)$ -module $J(C_f)_2$ could be easily described in terms of the (transitive) action of $\text{Gal}(f)$ on the set \mathfrak{R}_f of roots of f .

It turns out that if $\text{Gal}(f)$ contains \mathbf{A}_n then the Galois module $J(C_f)_2$ enjoys the following property ([17]):

(*): each subalgebra in $\text{End}_{\mathbf{F}_2}(J(C_f)_2)$ which contains the identity operator and is stable under the conjugation by Galois automorphisms either consists of scalars or coincides with $\text{End}_{\mathbf{F}_2}(J(C_f)_2)$.

Applying (*) to the subalgebra $\text{End}(J(C_f)) \otimes \mathbf{Z}/2\mathbf{Z}$, one concludes that it consists of scalars, i.e., $\text{End}(J(C_f))$ is a free abelian group of rank 1 and therefore coincides with \mathbf{Z} . (The case of $\text{End}(J(C)) \otimes \mathbf{Z}/2\mathbf{Z} = \text{End}_{\mathbf{F}_2}(J(C_f)_2)$ could not occur in characteristic zero.)

The proof of (*) was based on the well-known explicit description of $J(C_f)_2$ [14], [12] and elementary properties of \mathbf{A}_n and its simplest nontrivial representation in characteristic 2 of dimension $n - 1$ or $n - 2$ (depending on whether n is odd or even).

In this paper we study property (*) itself from the point of view of representation theory over \mathbf{F}_2 . Our results allow, in principle, to check the validity of (*) even if $\text{Gal}(f)$ does not contain \mathbf{A}_n . We prove that $\text{End}(J(C_f)) = \mathbf{Z}$ for an infinite series of $\text{Gal}(f) = \mathbf{L}_2(2^r) := \text{PSL}_2(\mathbf{F}_{2^r})$ and $n = 2^r + 1$ (with $r \geq 3$ and $\dim(J(C_f)) = 2^{r-1}$) or when $\text{Gal}(f)$ is the Suzuki group $\mathbf{Sz}(2^{2r+1})$ and $n = 2^{2(2r+1)} + 1$ (with $\dim(J(C_f)) = 2^{4r+1}$).

We refer the reader to [10], [11], [6], [7], [8], [17] for a discussion of known results about, and examples of, hyperelliptic jacobians without complex multiplication.

The paper is organized as follows. In §2 we state the main results and begin the discussion of linear representations for which an analogue of the property (*) holds true; we call such representations *very simple*. In §3 we prove that the very

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simplicity of the Galois module X_ℓ of points of prime order ℓ on an abelian variety X implies in characteristic zero that X does not have nontrivial endomorphisms. In §4 we remind basic facts about permutation groups and corresponding ordinary representations and modular representations over \mathbf{F}_2 . We use them in §5 in order to restate the main results as assertions about the very simplicity of certain permutation modules using an explicit description of points of order $\ell = 2$ on hyperelliptic jacobians. It turns out that all these permutation modules are Steinberg representations. In §6 we prove that the Steinberg representations are the only absolutely irreducible nontrivial representations (up to an isomorphism) over \mathbf{F}_2 for groups $\mathbf{L}_2(2^r)$ and $\mathbf{Sz}(2^{2r+1})$. In §7 we study very simple linear representations; in particular, we prove that all the (modular) Steinberg representations discussed in Section 6 are very simple. This ends the proof of main results.

2. MAIN RESULTS

Throughout this paper we assume that K is a field. We fix its algebraic closure K_a and write $\text{Gal}(K)$ for the absolute Galois group $\text{Aut}(K_a/K)$. If X is an abelian variety of dimension g defined over K then for each prime $\ell \neq \text{char}(K)$ we write X_ℓ for the kernel of multiplication by ℓ in $X(K_a)$. It is well-known that X_ℓ is a $2g$ -dimensional \mathbf{F}_ℓ -vector space provided with a natural structure of $\text{Gal}(K)$ -module. We write $\text{End}(X)$ for the ring of K_a -endomorphisms of X and $\text{End}^0(X)$ for the corresponding finite-dimensional \mathbf{Q} -algebra $\text{End}(X) \otimes \mathbf{Q}$.

The following notion plays a crucial role in this paper and will be discussed in detail in §7.

Definition 2.1. Let V be a vector space over a field \mathbf{F} , let G be a group and $\rho : G \rightarrow \text{Aut}_{\mathbf{F}}(V)$ a linear representation of G in V . We say that the G -module V is *very simple* if it enjoys the following property:

If $R \subset \text{End}_{\mathbf{F}}(V)$ be an \mathbf{F} -subalgebra containing the identity operator Id such that

$$\rho(\sigma)R\rho(\sigma)^{-1} \subset R \quad \forall \sigma \in G$$

then either $R = \mathbf{F} \cdot \text{Id}$ or $R = \text{End}_{\mathbf{F}}(V)$.

Remarks 2.2. (i) Clearly, the G -module V is very simple if and only if the corresponding $\rho(G)$ -module V is very simple.

(ii) Clearly, if V is very simple then the corresponding algebra homomorphism

$$\mathbf{F}[G] \rightarrow \text{End}_{\mathbf{F}}(V)$$

is surjective. Here $\mathbf{F}[G]$ stands for the group algebra of G . In particular, a very simple module is absolutely simple.

(iii) If G' is a subgroup of G and the G' -module V is very simple then the G -module V is also very simple.

(iv) Let G' be a normal subgroup of G . If V is a faithful very simple G -module then either $G' \subset \text{Aut}_{\mathbf{F}}(V)$ consists of scalars (i.e., lies in $\mathbf{F} \cdot \text{Id}$) or the G' -module V is also very simple.

Lemma 2.3. *Let X be an abelian variety of positive dimension g over K . Let ℓ be a prime different from $\text{char}(K)$. Assume that the $\text{Gal}(K)$ -module X_ℓ is very simple. Then either $\text{End}(X) = \mathbf{Z}$ or $\text{char}(K) > 0$ and X is a supersingular abelian variety.*

We prove Lemma 2.3 in §3.

Theorem 2.4. *Let K be a field with $\text{char}(K) \neq 2$, $f(x) \in K[x]$ an irreducible separable polynomial of degree $n \geq 5$. Let $\mathfrak{R} = \mathfrak{R}_f \subset K_a$ be the set of roots of f , let $K(\mathfrak{R}_f) = K(\mathfrak{R})$ be the splitting field of f and $\text{Gal}(f) := \text{Gal}(K(\mathfrak{R})/K)$ the Galois group of f , viewed as a subgroup of $\text{Perm}(\mathfrak{R})$. Let C_f be the hyperelliptic curve $y^2 = f(x)$. Let $J(C_f)$ be its jacobian, $\text{End}(J(C_f))$ the ring of K_a -endomorphisms of $J(C_f)$. Assume that n and $\text{Gal}(f)$ enjoy one of the following properties:*

- (i) $n = 2^m + 1 \geq 9$ and the Galois group $\text{Gal}(f)$ of f contains a subgroup isomorphic to $\mathbf{L}_2(2^m)$;
- (ii) For some positive integer k we have $n = 2^{2(2k+1)} + 1$ and the Galois group $\text{Gal}(f)$ of f is isomorphic to $\mathbf{Sz}(2^{2k+1})$;

Then:

- (a) The $\text{Gal}(K)$ -module $J(C)_2$ is very simple;
- (b) Either $\text{End}(J(C_f)) = \mathbf{Z}$ or $\text{char}(K) > 0$ and $J(C_f)$ is a supersingular abelian variety.

Remark 2.5. It follows from Lemma 2.3 that in order to prove Theorem 2.4, it suffices to check only the assertion a).

3. PROOF OF LEMMA 2.3

Recall that $\dim_{\mathbf{F}_\ell}(X_\ell) = 2g$. Since X is defined over K , one may associate with every $u \in \text{End}(X)$ and $\sigma \in \text{Gal}(K)$ an endomorphism ${}^\sigma u \in \text{End}(X)$ such that

$${}^\sigma u(x) = \sigma u(\sigma^{-1}x) \quad \forall x \in X(F_a).$$

Let us put

$$R := \text{End}(X) \otimes \mathbf{Z}/\ell\mathbf{Z} \subset \text{End}_{\mathbf{F}_\ell}(X_\ell).$$

Clearly, R satisfies all the conditions of Lemma 2.3. This implies that either $R = \mathbf{F}_\ell \cdot \text{Id}$ or $R = \text{End}_{\mathbf{F}_\ell}(X_\ell)$. If $\text{End}(X) \otimes \mathbf{Z}/\ell\mathbf{Z} = R = \mathbf{F}_\ell \cdot \text{Id}$ then the free abelian group $\text{End}(X)$ has rank 1 and therefore coincides with \mathbf{Z} . If $\text{End}(X) \otimes \mathbf{Z}/\ell\mathbf{Z} = R = \text{End}_{\mathbf{F}_\ell}(X_\ell)$ then the free abelian group $\text{End}(X)$ has rank $(2\dim(X))^2 = (2g)^2$ and therefore the \mathbf{Q} -algebra $\text{End}^0(X)$ has dimension $(2g)^2$.

Now Lemma 2.3 becomes an immediate corollary of the following assertion proven in [17] (see Lemma 3.1).

Lemma 3.1. *Let Y be an abelian variety of dimension g over an algebraically closed field K_a . Assume that the semisimple \mathbf{Q} -algebra $\text{End}^0(Y) = \text{End}(Y) \otimes \mathbf{Q}$ has dimension $(2g)^2$. Then $\text{char}(K_a) > 0$ and Y is supersingular.*

4. PERMUTATION GROUPS AND PERMUTATION MODULES

Let B be a finite set consisting of $n \geq 5$ elements. We write $\text{Perm}(B)$ for the group of permutations of B . A choice of ordering on B gives rise to an isomorphism

$$\text{Perm}(B) \cong \mathbf{S}_n.$$

Let G be a subgroup of $\text{Perm}(B)$. For each $b \in B$ we write G_b for the stabilizer of b in G ; it is a subgroup of G .

Remark 4.1. Assume that the action of G on B is transitive. It is well-known that each G_b is a subgroup of index n in G and all the G_b 's are conjugate one to another in G . Each conjugate of G_b in G is the stabilizer of a point in B . In addition, one may identify the G -set B with the set of cosets G/G_b with the standard action by G .

Let \mathbf{F} be a field. We write \mathbf{F}^B for the n -dimensional \mathbf{F} -vector space of maps $h : B \rightarrow \mathbf{F}$. The space \mathbf{F}^B is provided with a natural action of $\text{Perm}(B)$ defined as follows. Each $s \in \text{Perm}(B)$ sends a map $h : B \rightarrow \mathbf{F}$ into $sh : b \mapsto h(s^{-1}(b))$. The permutation module \mathbf{F}^B contains the $\text{Perm}(B)$ -stable hyperplane

$$(\mathbf{F}^B)^0 = \{h : B \rightarrow \mathbf{F} \mid \sum_{b \in B} h(b) = 0\}$$

and the $\text{Perm}(B)$ -invariant line $\mathbf{F} \cdot 1_B$ where 1_B is the constant function 1. The quotient $\mathbf{F}^B/(\mathbf{F}^B)^0$ is a trivial 1-dimensional $\text{Perm}(B)$ -module.

Clearly, $(\mathbf{F}^B)^0$ contains $\mathbf{F} \cdot 1_B$ if and only if $\text{char}(\mathbf{F})$ divides n . If this is *not* the case then there is a $\text{Perm}(B)$ -invariant splitting

$$\mathbf{F}^B = (\mathbf{F}^B)^0 \oplus \mathbf{F} \cdot 1_B.$$

Clearly, \mathbf{F}^B and $(\mathbf{F}^B)^0$ carry natural structures of G -modules. Their characters depend only on characteristic of \mathbf{F} .

Let us consider the case of $\mathbf{F} = \mathbf{Q}$. Then the character of \mathbf{Q}^B sends each $g \in G$ into the number of fixed points of g ([15], ex. 2.2, p. 12); it is called the *permutation character*. Let us denote by $\chi = \chi_B : G \rightarrow \mathbf{Q}$ the character of $(\mathbf{Q}^B)^0$. It is known that the $\mathbf{Q}[G]$ -module $(\mathbf{Q}^B)^0$ is absolutely simple if and only if G acts doubly transitively on B ([15], ex. 2.6, p. 17). Clearly, $1 + \chi$ is the permutation character.

Now, let us consider the case of $\mathbf{F} = \mathbf{F}_2$. It is well-known that one may view \mathbf{F}_2^B as the \mathbf{F}_2 -vector space of *all* subsets of B with symmetric difference as a sum. Namely, a subset T corresponds to its characteristic function $\chi_T : B \rightarrow \{0, 1\} = \mathbf{F}_2$ and a function $h : B \rightarrow \mathbf{F}_2$ corresponds to its support $\text{supp}(h) = \{x \in B \mid h(x) = 1\}$. Under this identification each $s \in G \subset \text{Perm}(B)$ sends T into $s(T) = \{s(b) \mid b \in T\}$.

Under this identification the hyperplane $(\mathbf{F}_2^B)^0$ corresponds to the \mathbf{F}_2 -vector space of *all* subsets of B of *even* cardinality with symmetric difference as a sum.

If n is even then let us define the $\text{Perm}(B)$ -module

$$Q_B := (\mathbf{F}_2^B)^0 / (\mathbf{F}_2 \cdot 1_B).$$

If n is odd then let us put

$$Q_B := (\mathbf{F}_2^B)^0.$$

When n is even, the quotient Q_B corresponds to the $n - 2$ -dimensional \mathbf{F}_2 -vector space of *all* subsets of B of *even* cardinality with symmetric difference as a sum where each subset $T \subset B$ of even cardinality is identified with its complement $B \setminus T$.

Remark 4.2. Clearly, $\dim_{\mathbf{F}_2}(Q_B) = n - 1$ if n is odd and $\dim_{\mathbf{F}_2}(Q_B) = n - 2$ if n is even. In both cases Q_B is a faithful G -module.

Let $G^{(2)}$ be the set of 2-regular elements of G . Clearly, the Brauer character of the G -module \mathbf{F}_2^B coincides with the restriction of $1 + \chi_B$ to $G^{(2)}$. This implies easily

that the Brauer character of the G -module $(\mathbf{F}_2^B)^0$ coincides with the restriction of χ_B to $G^{(2)}$.

Remark 4.3. Let us denote by $\phi_B = \phi$ the Brauer character of the G -module Q_B . One may easily check that ϕ_B coincides with the restriction of χ_B to $G^{(2)}$ if n is odd and with the restriction of $\chi_B - 1$ to $G^{(2)}$ if n is even.

Remark 4.4. Assume that $n = \#(B)$ is even. Let us choose $b \in B$ and let $G' := G_b$ and $B' = B \setminus \{b\}$. Then $n' = \#(B') = n - 1$ is odd and there is a canonical isomorphism of G' -modules $Q_{B'} \cong Q_B$ defined as follows. First, there is a natural G' -equivariant embedding $\mathbf{F}_2^{B'} \subset \mathbf{F}_2^B$ which could be obtained by extending each $h : B' \rightarrow \mathbf{F}_2$ to B by letting $h(b) = 0$. Second, this embedding identifies $(\mathbf{F}_2^{B'})^0$ with a hyperplane of $(\mathbf{F}_2^B)^0$ which does not contain 1_B . Now the desired isomorphism is given by the composition

$$Q_{B'} = (\mathbf{F}_2^{B'})^0 \subset (\mathbf{F}_2^B)^0 \rightarrow (\mathbf{F}_2^B)^0 / (\mathbf{F}_2 \cdot 1_B) = Q_B.$$

This implies that if the G' -module $Q_{B'}$ is very simple then the G -module Q_B is also very simple.

Remark 4.5. Assume that G acts on B doubly transitively, $\#(B)$ is odd and $\#(B) - 1 = \dim_{\mathbf{Q}}((\mathbf{Q}_B)^0)$ coincides with the largest power of 2 dividing $\#(G)$. Then it follows from a theorem of Brauer-Nesbitt ([15], Sect. 16.4, pp. 136–137 ; [4], p. 249) that Q_B is an absolutely simple $\mathbf{F}_2[G]$ -module. In particular, Q_B is (the reduction of) the Steinberg representation [4].

5. POINTS OF ORDER 2 ON HYPERELLIPTIC JACOBIANS

We keep all notations of Section 2. In addition, we assume that K is a field of characteristic different from 2. Let C be a hyperelliptic curve over K defined by an equation $y^2 = f(x)$ where $f(x) \in K[x]$ is a polynomial of degree $n \geq 5$ without multiple roots. The rational function $x \in K(C)$ defines a canonical double cover $\pi : C \rightarrow \mathbf{P}^1$. Let $B' \subset C(K_a)$ be the set of ramification points of π (Weierstraß points). Clearly, the restriction of π to B' is an injective map $\pi : B' \hookrightarrow \mathbf{P}^1(K_a)$, whose image is either the set $\mathfrak{R} = \mathfrak{R}_f$ of roots of f if n is even or the disjoint union of ∞ and \mathfrak{R} if n is odd. By abuse of notation, we also denote by ∞ the ramification point lying above ∞ if n is odd and by ∞_1 and ∞_2 two unramified points lying above ∞ if n is even. Clearly, if n is odd then $\infty \in C(K)$. If n is even then the 2-element set $\{\infty_1, \infty_2\}$ is stable under the action of $\text{Gal}(K)$.

Let us put

$$B = \{(\alpha, 0) \mid f(\alpha) = 0\} \subset C(K_a).$$

Then π defines a bijection between B and \mathfrak{R} which commutes with the action of $\text{Gal}(K)$. If n is even then B coincides with B' . In the case of odd n the set B' is the disjoint union of B and ∞ .

Theorem 5.1. *Suppose n is an integer which is greater than or equal to 5. Suppose $f(x) \in K[x]$ is a separable polynomial of degree n , $\mathfrak{R} \subset K_a$ the set of roots of f , let $K(\mathfrak{R})$ be the splitting field of f and $\text{Gal}(f) := \text{Gal}(K(\mathfrak{R})/K)$ the Galois group of f .*

Suppose C is the hyperelliptic curve $y^2 = f(x)$ of genus $g = \lfloor \frac{n-1}{2} \rfloor$ over K . Suppose $J(C)$ is the jacobian of C and $J(C)_2$ is the group of its points of order 2, viewed as a $2g$ -dimensional \mathbf{F}_2 -vector space provided with the natural action of

$\text{Gal}(K)$. Then the homomorphism $\text{Gal}(K) \rightarrow \text{Aut}_{\mathbf{F}_2}(J(C)_2)$ factors through the canonical surjection $\text{Gal}(K) \twoheadrightarrow \text{Gal}(K(\mathfrak{R})/K) = \text{Gal}(f)$ and the $\text{Gal}(f)$ -modules $J(C)_2$ and $Q_{\mathfrak{R}}$ are isomorphic. In particular, the $G(K)$ -module $J(C)_2$ is very simple if and only if the $\text{Gal}(f)$ -module Q_B is very simple.

Remark 5.2. Clearly, $\text{Gal}(K)$ acts on B through the canonical surjective homomorphism $\text{Gal}(K) \twoheadrightarrow \text{Gal}(f)$, because all points of B are defined over $K(\mathfrak{R})$ and the natural homomorphism $\text{Gal}(f) \rightarrow \text{Perm}(B)$ is injective. Clearly, $\pi : B \rightarrow \mathfrak{R}$ is a bijection of $\text{Gal}(f)$ -sets. This implies easily that the $\text{Gal}(f)$ -modules Q_B and $Q_{\mathfrak{R}}$ are isomorphic. So, in order to prove Theorem 5.1 it suffices to check that the $\text{Gal}(K)$ -modules Q_B and $J(C)_2$ are isomorphic.

Proof of Theorem 5.1. Here is a well-known explicit description of the group $J(C)_2$ of points of order 2 on $J(C)$. Let us denote by L the K -divisor $2(\infty)$ on C if n is odd and the K -divisor $(\infty_1) + (\infty_2)$ if n is even. In both cases L is an effective divisor of degree 2. Namely, let $T \subset B'$ be a subset of even cardinality. Then ([14], Ch. IIIa, Sect. 2, Lemma 2.4; [12], pp. 190–191; see also [11]) the divisor $e_T = \sum_{P \in T} (P) - \frac{\#(T)}{2}L$ on C has degree 0 and $2e_T$ is principal. If T_1, T_2 are two subsets of even cardinality in B' then the divisors e_{T_1} and e_{T_2} are linearly equivalent if and only if either $T_1 = T_2$ or $T_2 = B' \setminus T_1$. Also, if $T = T_1 \Delta T_2$ then the divisor e_T is linearly equivalent to $e_{T_1} + e_{T_2}$. Hereafter we use the symbol Δ for the symmetric difference of two sets. Counting arguments imply easily that each point of $J(C)_2$ is the class of e_T for some T . We know that such a choice is not unique. However, in the case of odd n if we demand that T does not contain ∞ then such a choice always exists and unique. This observation leads to a canonical group isomorphism

$$Q_B = (\mathbf{F}_2^B)^0 \cong J(C)_2, \quad T \mapsto \text{cl}(e_T)$$

in the case of odd n . Here cl stands for the linear equivalence class of a divisor. In the case of even n we are still able to define a canonical surjective group homomorphism

$$(\mathbf{F}_2^B)^0 \rightarrow J(C)_2, \quad T \mapsto \text{cl}(e_T)$$

and one may easily check that the kernel of this map is the line generated by the set B , i.e., the line generated by the constant function 1_B . This gives rise to the injective homomorphism

$$Q_B = (\mathbf{F}_2^B)^0 / (\mathbf{F}_2 \cdot 1_B) \rightarrow J(C)_2,$$

which is an isomorphism, by counting arguments. So, in both (odd and even) cases we get a canonical isomorphism $Q_B \cong J(C)_2$, which obviously commutes with the actions of $\text{Gal}(K)$. In other words, we constructed an isomorphism of $\text{Gal}(K)$ -modules Q_B and $J(C)_2$. In light of Remark 5.2, this ends the proof of Theorem 5.1. \square

Combining Theorem 5.1 and Lemma 2.3 (for $\ell = 2$), we obtain the following corollary.

Corollary 5.3. *Let K be a field with $\text{char}(K) \neq 2$, K_a its algebraic closure, $f(x) \in K[x]$ an irreducible separable polynomial of degree $n \geq 5$. Let $\mathfrak{R} = \mathfrak{R}_f \subset K_a$ be the set of roots of f , let $K(\mathfrak{R}_f) = K(\mathfrak{R})$ be the splitting field of f and $\text{Gal}(f) := \text{Gal}(K(\mathfrak{R})/K)$ the Galois group of f , viewed as a subgroup of $\text{Perm}(\mathfrak{R})$. Let C_f be the hyperelliptic curve $y^2 = f(x)$. Let $J(C_f)$ be its jacobian, $\text{End}(J(C_f))$ the*

ring of K_a -endomorphisms of $J(C_f)$. Assume that the $\text{Gal}(f)$ -module $Q_{\mathfrak{A}}$ is very simple. Then either $\text{End}(J(C_f)) = \mathbf{Z}$ or $\text{char}(K) > 0$ and $J(C_f)$ is a supersingular abelian variety.

Notice that in order to prove Theorem 2.4, it suffices to check the following statement.

Theorem 5.4. *Let n be a positive integer, B a n -element set, $H \subset \text{Perm}(B)$ a permutation group. Assume that (n, H) enjoy one of the following properties:*

- (i) $n = 2^m + 1 \geq 9$ and H contains a subgroup isomorphic to $\mathbf{L}_2(2^m)$;
- (ii) For some positive integer k we have $n = 2^{2(2k+1)} + 1$ and H contains a subgroup isomorphic to $\mathbf{Sz}(2^{2k+1})$;

Then the H -module Q_B is very simple.

Proof of Theorem 2.4 modulo Theorem 5.4. Let us put

$$n = \deg(f), B = \mathfrak{A}, H = \text{Gal}(f).$$

It follows from Theorem 5.4 that the $\text{Gal}(f)$ -module $\mathbf{Q}_{\mathfrak{A}}$ is very simple. Now the result follows readily from Corollary 5.3. \square

We prove Theorem 5.4 at the end of §7.

6. STEINBERG REPRESENTATION

In this section we prove that the Steinberg representation is the only nontrivial absolutely irreducible representation over \mathbf{F}_2 (up to an isomorphism) of groups $\mathbf{L}_2(2^m)$ and $\mathbf{Sz}(2^{2k+1})$. We refer to [4] for basic properties of Steinberg representations.

Let us fix an algebraic closure of \mathbf{F}_2 and denote it by \mathcal{F} . We write $\phi : \mathcal{F} \rightarrow \mathcal{F}$ for the Frobenius automorphism $x \mapsto x^2$. Let $q = 2^m$ be a positive integral power of two. Then the subfield of invariants of $\phi^m : \mathcal{F} \rightarrow \mathcal{F}$ is a finite field \mathbf{F}_q consisting of q elements. Let q' be an integral positive power of q . If d is a positive integer and i is a non-negative integer then for each matrix $u \in \text{GL}_d(\mathcal{F})$ we write $u^{(i)}$ for the matrix obtained by raising each entry of u to the 2^i th power.

Recall that an element $\alpha \in \mathbf{F}_q$ is called *primitive* if $\alpha \neq 0$ and has multiplicative order $q - 1$ in the cyclic multiplicative group \mathbf{F}_q^* .

Lemma 6.1. *Let $q > 2$, let d be a positive integer and let G be a subgroup of $\text{GL}_d(\mathbf{F}_{q'})$. Assume that there exists an element $u \in G \subset \text{GL}_d(\mathbf{F}_{q'})$, whose trace α lies in \mathbf{F}_q^* and has multiplicative order $q - 1$. Let $V_0 = \mathcal{F}^d$ and let $\rho_0 : G \subset \text{GL}_d(\mathbf{F}_{q'}) \subset \text{GL}_d(\mathcal{F}) = \text{Aut}_{\mathcal{F}}(V_0)$ be the natural d -dimensional representation of G over \mathcal{F} . For each positive integer $i < m$ we define a d -dimensional \mathcal{F} -representation*

$$\rho_i : G \rightarrow \text{Aut}(V_i)$$

as the composition of

$$G \hookrightarrow \text{GL}_d(\mathbf{F}_{q'}), \quad x \mapsto x^{(i)}$$

and the inclusion map

$$\text{GL}_d(\mathbf{F}_{q'}) \subset \text{GL}_d(\mathcal{F}) \cong \text{Aut}_{\mathcal{F}}(V_i).$$

Let S be a subset of $\{0, 1, \dots, m-1\}$. Let us define a $d^{\#(S)}$ -dimensional \mathcal{F} -representation ρ_S of G as the tensor product of representations ρ_i for all $i \in S$. If S is a proper subset of $\{0, 1, \dots, m-1\}$ then there exists an element $u \in G$ such that the trace of

$\rho_S(u)$ does not belong to \mathbf{F}_2 . In particular, ρ_S could not be obtained by extension of scalars to \mathcal{F} from a representation of G over \mathbf{F}_2 .

Proof. Clearly,

$$\mathrm{tr}(\rho_i(u)) = (\mathrm{tr}(\rho_0(u)))^{2^i} \quad \forall u \in G.$$

This implies easily that

$$\mathrm{tr}(\rho_S(u)) = \prod_{i \in S} \mathrm{tr}(\rho_i(u)) = (\mathrm{tr}(\rho_0(u)))^M$$

where $M = \sum_{i \in S} 2^i$. Since S is a proper subset of $\{0, 1, \dots, m-1\}$, we have

$$0 < M < \sum_{i=0}^{m-1} 2^i = 2^m - 1 = \#(\mathbf{F}_q^*).$$

Recall that there exists $u \in G$ such that $\alpha = \mathrm{tr}(\rho_0(u))$ lies in \mathbf{F}_q^* and the exact multiplicative order of α is $q-1 = 2^m - 1$.

This implies that $0 \neq \alpha^M \neq 1$. Since $\mathbf{F}_2 = \{0, 1\}$, we conclude that $\alpha^M \notin \mathbf{F}_2$. Therefore $\mathrm{tr}(\rho_S(u)) = (\mathrm{tr}(\rho_0(u)))^M = \alpha^M \notin \mathbf{F}_2$. \square

Theorem 6.2. *Let $q \geq 8$ be a power of 2 and $G = \mathbf{L}_2(q) = \mathrm{PSL}_2(\mathbf{F}_q) = \mathrm{SL}_2(\mathbf{F}_q)$. Let $\rho : G \rightarrow \mathrm{Aut}(V)$ be an absolutely irreducible faithful representation of G over \mathcal{F} . If the trace map $\mathrm{tr}_\rho : G \rightarrow \mathcal{F}$ takes on values in \mathbf{F}_2 then $\dim_{\mathcal{F}}(V) = q$. In particular, ρ is the Steinberg representation of G .*

Proof. Let us put $q' = q$. We have

$$G = \mathrm{SL}_2(\mathbf{F}_q) \subset \mathrm{GL}_2(\mathbf{F}_q).$$

Clearly, for each $\alpha \in \mathbf{F}_q$ one may find a 2×2 matrix with determinant 1 and trace α . This implies that G satisfies the conditions of Lemma 6.1.

The construction described in Lemma 6.1 allows us to construct a $d^{\#(S)}$ -dimensional \mathcal{F} -representation ρ_S of G for each subset S of $\{0, 1, \dots, m-1\}$. It is well-known ([1], pp. 588-589) that ρ_S 's exhaust the list of all absolutely irreducible \mathcal{F} -representations of $G = \mathrm{SL}_2(\mathbf{F}_q)$ and therefore ρ is isomorphic to ρ_S for some S . It follows from Lemma 6.1 that either S is empty or $S = \{0, 1, \dots, m-1\}$. The case of empty S corresponds to the trivial 1-dimensional representation. Therefore $S = \{0, 1, \dots, m-1\}$ and ρ is $2^m = q$ -dimensional. \square

Suppose $m = 2k + 1 \geq 3$ is an odd integer. Let $q = 2^m = 2^{2k+1}$ and $d = 4$. Recall ([3], pp. 182-194) that the *Suzuki group* $\mathbf{Sz}(q)$ is the subgroup of $\mathrm{GL}_4(\mathbf{F}_q)$ generated by the matrices $S(a, b), M(\lambda), T$ defined as follows. For each $a, b \in \mathbf{F}_q$ the matrix $S(a, b)$ is defined by

$$S(a, b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & a^{2^{k+1}} & 1 & 0 \\ a^{2^{k+1}+2} + ab + b^{2^{k+1}} & a^{2^{k+1}+1} + b & a & 1 \end{pmatrix}$$

and for each $\lambda \in \mathbf{F}_q^*$ the matrix $M(\lambda)$ is defined by

$$M(\lambda) = \begin{pmatrix} \lambda^{1+2^k} & 0 & 0 & 0 \\ 0 & \lambda^{2^k} & 0 & 0 \\ 0 & 0 & \lambda^{-2^k} & 0 \\ 0 & 0 & 0 & \lambda^{1+2^k} \end{pmatrix}.$$

The matrix T is defined by

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that the trace of $S(0, b)T$ is $b^{2^{k+1}}$. This implies easily that for each $\alpha \in \mathbf{F}_q$ one may find an element of $\mathbf{Sz}(q) \subset \mathrm{GL}_4(\mathbf{F}_q)$ with trace α . This implies that

$$G = \mathbf{Sz}(q) \subset \mathrm{GL}_4(\mathbf{F}_q)$$

satisfies the conditions of Lemma 6.1. Notice also that $\#(\mathbf{Sz}(q)) = (q^2 + 1)q^2(q - 1)$ ([3], p. 187).

Theorem 6.3. *Let $\rho : \mathbf{Sz}(q) \rightarrow \mathrm{Aut}(V)$ be an absolutely irreducible faithful representation of $\mathbf{Sz}(q)$ over \mathcal{F} . If the trace map $\mathrm{tr}_\rho : \mathbf{Sz}(q) \rightarrow \mathcal{F}$ takes on values in \mathbf{F}_2 then $\dim_{\mathcal{F}}(V) = q^2$. In particular, ρ is the Steinberg representation of G .*

Proof. Let us put $q' = q$. We know that $G = \mathbf{Sz}(q) \subset \mathrm{GL}_4(\mathbf{F}_q)$ satisfies the conditions of Lemma 6.1.

The construction described in Lemma 6.1 allows us to construct a $4^{\#(S)}$ -dimensional \mathcal{F} -representation ρ_S of G for each subset S of $\{0, 1, \dots, m-1\}$. It is known ([9], pp. 56–57) that ρ_S 's exhaust the list of all absolutely irreducible \mathcal{F} -representations of $G = \mathrm{SL}_2(\mathbf{F}_q)$ and therefore ρ is isomorphic to ρ_S for some S . It follows from Lemma 6.1 that either S is empty or $S = \{0, 1, \dots, m-1\}$. The case of empty S corresponds to trivial 1-dimensional representation. Therefore $S = \{0, 1, \dots, m-1\}$ and ρ is $4^m = q^2$ -dimensional. \square

Remark 6.4. Assume that in the case 5.4(i) (resp. 5.4(ii)) that $H = \mathbf{L}_2(2^m)$ (resp. $\mathbf{Sz}(2^{2k+1})$). It follows from Remark 4.5 that Q_B is the Steinberg representation of H .

7. VERY SIMPLE REPRESENTATIONS

Examples 7.1. (i) If $\dim(V) = 1$ then V is always very simple.

(ii) Assume that there exist G -modules V_1 and V_2 such that $\dim(V_1) > 1, \dim(V_2) > 1$ and the G -module V is isomorphic to $V_1 \otimes_{\mathbf{F}} V_2$. Then V is *not* very simple. Indeed, the subalgebra

$$R = \mathrm{End}_{\mathbf{F}}(V_1) \otimes \mathbf{F} \cdot \mathrm{Id}_{V_2} \subset \mathrm{End}_{\mathbf{F}}(V_1) \otimes_{\mathbf{F}} \mathrm{End}_{\mathbf{F}}(V_2) = \mathrm{End}_{\mathbf{F}}(V)$$

is stable under the conjugation by elements of G but coincides neither with $\mathbf{F} \cdot \mathrm{Id}$ nor with $\mathrm{End}_{\mathbf{F}}(V)$. (Here Id_{V_2} stands for the identity operator in V_2 .)

(ii)bis Let $X \rightarrow G$ be a central extension of G . Assume that there exist X -modules V_1 and V_2 such that $\dim(V_1) > 1, \dim(V_2) > 1$ and V , viewed as X -module, is isomorphic to $V_1 \otimes_{\mathbf{F}} V_2$. Then V is *not* very simple as an X -module. Since X and G have the same images in $\mathrm{Aut}_{\mathbf{F}}(V)$, the G -module V is also not very simple.

(iii) Assume that there exists a subgroup G' in G of finite index $m > 1$ and a G' -module V' such that the $\mathbf{F}[G]$ -module V is *induced* by the $\mathbf{F}[G']$ -module V' . (In particular, m must divide $\dim(V)$.) Then V is *not* very simple. Indeed, one may view W as a G' -submodule of V such that V coincides with the direct sum $\oplus_{\sigma \in G/G'} \sigma W$. Let $R = \oplus_{\sigma \in G/G'} \mathrm{End}_{\mathbf{F}}(\sigma W)$ be the algebra of all

operators sending each σW into itself. Then R is stable under the conjugation by elements of G but coincides neither with $\mathbf{F} \cdot \text{Id}$ nor with $\text{End}_{\mathbf{F}}(V)$.

Example 7.2. Let $n \geq 5$ be an integer, B a n -element set. Suppose G is either $\text{Perm}(B) \cong \mathbf{S}_n$ or the only subgroup in $\text{Perm}(B)$ of index 2 (isomorphic to \mathbf{A}_n). Then the G -module Q_B is very simple. If n is odd then this assertion is proven in [17], Th. 4.1. If n is even then $n \geq 6$, $n' = n - 1 \geq 5$ is odd and the result follows from the odd case combined with Remarks 2.2 and 4.4.

Remarks 7.3. Assume that there exist G -modules V_1 and V_2 such that $\dim(V_1) > 1$, $\dim(V_2) > 1$ and the G -module V is isomorphic to $V_1 \otimes_{\mathbf{F}} V_2$.

- (i) If V is simple then both V_1 and V_2 are also simple. Indeed, if say, V' is a proper G -stable subspace in V_1 then $V' \otimes_{\mathbf{F}} V_2$ is a proper G -stable subspace in $V_1 \otimes_{\mathbf{F}} V_2 = V$.
- (ii) If V is absolutely simple then both V_1 and V_2 are also absolutely simple. Indeed, assume that say, $R_1 := \text{End}_G(V_1)$ has \mathbf{F} -dimension greater than 1. Then $\text{End}_G(V) = \text{End}_G(V_1 \otimes_{\mathbf{F}} V_2)$ contains $R_1 \otimes \text{Id}_{V_2} \cong R_1$ and therefore also has dimension greater than 1.

Lemma 7.4. Let H be a group, \mathbf{F} a field and V a simple $\mathbf{F}[H]$ -module of finite \mathbf{F} -dimension N . Let $R \subset \text{End}_{\mathbf{F}}(V)$ be an \mathbf{F} -subalgebra containing the identity operator Id and such that

$$uRu^{-1} \subset R \quad \forall u \in H.$$

Then:

- (i) The faithful R -module V is semisimple.
- (ii) Either the R -module V is isotypic or there exists a subgroup $H' \subset H$ of index r dividing N and a H' -module V' of finite \mathbf{F} -dimension N/r such that $r > 1$ and the H -module V is induced by V' . In addition, if $\mathbf{F} = \mathbf{F}_2$ then $r < N$.

Proof. We may assume that $N > 1$. Clearly, V is a faithful R -module and

$$uRu^{-1} = R \quad \forall u \in H.$$

Step 1. V is a semisimple R -module. Indeed, let $U \subset V$ be a simple R -submodule. Then $U' = \sum_{s \in H} sU$ is a non-zero H -stable subspace in V and therefore must coincide with V . On the other hand, each sU is also a R -submodule in V , because $s^{-1}Rs = R$. In addition, if $W \subset sU$ is an R -submodule then $s^{-1}W$ is an R -submodule in U , because

$$Rs^{-1}W = s^{-1}sRs^{-1}W = s^{-1}RW = s^{-1}W.$$

Since U is simple, $s^{-1}W = \{0\}$ or U . This implies that sU is also simple. Hence $V = U'$ is a sum of simple R -modules and therefore is a semisimple R -module.

Step 2. The R -module V is either isotypic or induced. Indeed, let us split the semisimple R -module V into the direct sum

$$V = V_1 \oplus \cdots \oplus V_r$$

of its isotypic components. Dimension arguments imply that $r \leq \dim(V) = N$. It follows easily from the arguments of the previous step that for each isotypic component V_i its image sV_i is an isotypic R -submodule for each $s \in H$ and therefore is

contained in some V_j . Similarly, $s^{-1}V_j$ is an isotypic submodule obviously containing V_i . Since V_i is the isotypic component, $s^{-1}V_j = V_i$ and therefore $sV_i = V_j$. This means that s permutes the V_i ; since V is H -simple, H permutes them transitively.

This implies that all V_i have the same dimension N/r and therefore r divides $\dim(V) = N$. Let $H' = H_i$ be the stabilizer of V_i in H , i.e.

$$H_i = \{s \in H \mid sV_i = V_i\}.$$

The transitivity of the action of H on V_j s implies that $[H : H_i] = r$.

If $r = 1$ then $H = H' = H_i$. This means that $sV_i = V_i$ for all $s \in H$ and $V = V_i$ is isotypic.

Assume that $r > 1$ and consider the H' -module $W = V_i$. Clearly, $[H : H'] = [H : H_i] = r$ divides N and the H -module V is induced by W .

Step 3. Assume that $r = N$ and $\mathbf{F} = \mathbf{F}_2$. Then each V_i is one-dimensional and contains exactly one non-zero vector say, v_i . Then the sum $\sum_{i=1}^N v_i$ is a non-zero H -invariant vector which contradicts the simplicity of the H -module V . \square

Theorem 7.5. *Suppose H is a group and*

$$\rho : H \rightarrow \text{Aut}_{\mathbf{F}_2}(V)$$

is an absolutely simple $\mathbf{F}_2[H]$ -module of finite dimension N . Suppose there exists an \mathbf{F}_2 -subalgebra $R \subset \text{End}_{\mathbf{F}_2}(V)$ containing the identity operator Id and such that

$$uRu^{-1} \subset R \quad \forall u \in H.$$

Assume, in addition, that H does not have nontrivial cyclic quotients of order dividing N . If the R -module V is isotypic then there exist $\mathbf{F}_2[H]$ -modules V_1 and V_2 such that V , viewed as H -module, is isomorphic to $V_1 \otimes_{\mathbf{F}_2} V_2$ and the image of $R \subset \text{End}_{\mathbf{F}_2}(V)$ under the induced isomorphism

$$\text{End}_{\mathbf{F}_2}(V) = \text{End}_{\mathbf{F}_2}(V_1 \otimes_{\mathbf{F}_2} V_2) = \text{End}_{\mathbf{F}_2}(V_1) \otimes_{\mathbf{F}_2} \text{End}_{\mathbf{F}_2}(V_2)$$

coincides with $\text{End}_{\mathbf{F}_2}(V_1) \otimes \text{Id}_{V_2}$. In particular, if both V_1 and V_2 have dimension greater than 1 then the H -module V is not very simple.

Proof. Since V is isotypic, there exist a simple R -module W , a positive integer d and an isomorphism

$$\psi : V \cong W^d$$

of R -modules. Let us put

$$V_1 = W, \quad V_2 = \mathbf{F}_2^d.$$

The isomorphism ψ gives rise to the isomorphism of \mathbf{F}_2 -vector spaces

$$V = W^d = W \otimes_{\mathbf{F}_2} \mathbf{F}_2^d = V_1 \otimes_{\mathbf{F}_2} V_2.$$

We have

$$d \cdot \dim(W) = \dim(V) = N.$$

Clearly, $\text{End}_R(V)$ is isomorphic to the matrix algebra $\text{Mat}_d(\text{End}_R(W))$ of size d over $\text{End}_R(W)$.

Let us put

$$k = \text{End}_R(W).$$

Since W is simple, k is a finite-dimensional division algebra over \mathbf{F}_2 . Therefore k must be a finite field.

We have

$$\text{End}_R(V) \cong \text{Mat}_d(k).$$

Clearly, $[k : \mathbf{F}_2]$ divides $\dim_{\mathbf{F}_2}(W)$ and therefore divides $\dim_{\mathbf{F}_2}(V) = N$. Clearly, $\text{Aut}(k/\mathbf{F}_2)$ is always a cyclic group of order $[k : \mathbf{F}_2]$ and therefore has order dividing N .

Clearly, $\text{End}_R(V) \subset \text{End}_{\mathbf{F}_2}(V)$ is stable under the adjoint action of H . This induces a homomorphism

$$\alpha : H \rightarrow \text{Aut}_{\mathbf{F}_2}(\text{End}_R(V)) = \text{Aut}_{\mathbf{F}_2}(\text{Mat}_d(k)).$$

Since k is the center of $\text{Mat}_d(k)$, it is stable under the action of H , i.e., we get a homomorphism $H \rightarrow \text{Aut}(k/\mathbf{F}_2)$, which must be trivial, since H is perfect and $\text{Aut}(k/\mathbf{F}_2)$ is a cyclic group of order dividing N and therefore the kernel of the homomorphism must coincide with H . This implies that the center k of $\text{End}_R(V)$ commutes with H . Since $\text{End}_H(V) = \mathbf{F}_2$, we have $k = \mathbf{F}_2$. This implies that $\text{End}_R(V) \cong \text{Mat}_d(\mathbf{F}_2)$ and one may rewrite α as

$$\alpha : H \rightarrow \text{Aut}_{\mathbf{F}_2}(\text{Mat}_d(\mathbf{F}_2)) = \text{Aut}(\text{End}_{\mathbf{F}_2}(V_2)) = \text{Aut}_{\mathbf{F}_2}(V_2)/\mathbf{F}_2^* = \text{Aut}_{\mathbf{F}_2}(V_2).$$

It follows from the Jacobson density theorem that $R = \text{End}_{\mathbf{F}_2}(W) \cong \text{Mat}_m(\mathbf{F}_2)$ with $dm = N$.

The adjoint action of H on R gives rise to a homomorphism

$$\beta : H \rightarrow \text{Aut}_{\mathbf{F}_2}(\text{End}_{\mathbf{F}_2}(W)) = \text{Aut}_{\mathbf{F}_2}(W)/\mathbf{F}_2^* = \text{Aut}_{\mathbf{F}_2}(W).$$

Clearly, α and β provide V_2 and V_1 respectively with the structure of H -modules. Notice that

$$R = \text{End}_{\mathbf{F}_2}(V_1) = \text{End}_{\mathbf{F}_2}(V_1) \otimes \text{Id}_{V_2} \subset \text{End}_{\mathbf{F}_2}(V_1) \otimes_{\mathbf{F}_2} \text{End}_{\mathbf{F}_2}(V_2) = \text{End}_{\mathbf{F}_2}(V).$$

Now our task boils down to comparison of the structures of H -module on $V = V_1 \otimes_{\mathbf{F}_2} V_2$ defined by ρ and $\beta \otimes \alpha$ respectively. I claim that

$$\rho(g) = \beta(g) \otimes \alpha(g) \quad \forall g \in H.$$

Indeed, notice that the conjugation by $\rho(g)$ in $\text{End}_{\mathbf{F}_2}(V) = \text{End}_{\mathbf{F}_2}(V_1 \otimes_{\mathbf{F}_2} V_2)$ leaves stable $R = \text{End}_{\mathbf{F}_2}(V_1) \otimes_{\mathbf{F}_2} \text{Id}_{V_2}$ and coincides on R with the conjugation by $\alpha(g) \otimes \text{Id}_{V_2}$. Since the centralizer of $\text{End}_{\mathbf{F}_2}(V_1) \otimes \text{Id}_{V_2}$ in

$$\text{End}_{\mathbf{F}_2}(V) = \text{End}_{\mathbf{F}_2}(V_1) \otimes_{\mathbf{F}_2} \text{End}_{\mathbf{F}_2}(V_2)$$

coincides with $\text{Id}_{V_1} \otimes \text{End}_{\mathbf{F}_2}(V_2)$, there exists $u \in \text{Aut}_{\mathbf{F}_2}(V_2)$ such that

$$\rho(g) = \beta(g) \otimes u.$$

Since the conjugation by $\rho(g)$ leaves stable the centralizer of R , i.e. $\text{Id}_{V_1} \otimes \text{End}_{\mathbf{F}_2}(V_2)$ and coincides on it with the conjugation by $\text{Id}_{V_1} \otimes \alpha(g)$, there exists a non-zero constant $\gamma \in \mathbf{F}_2^*$ such that $u = \gamma\beta(g)$. This implies that

$$\rho(g) = \beta(g) \otimes u = \gamma \cdot \beta(g) \otimes \alpha(g).$$

Now one has only to recall that $\mathbf{F}_2^* = \{1\}$ and therefore $\gamma = 1$. □

Remark 7.6. In the notations of Th. 7.5 the H -modules V_1 and V_2 must be absolutely simple. It follows easily from Remarks 7.3.

Lemma 7.4 and Theorem 7.5 together with Remark 7.6 imply easily the following criterion of very simplicity over \mathbf{F}_2 .

Theorem 7.7. *Let H be a group and V be a $\mathbf{F}_2[H]$ -module of finite dimension N over \mathbf{F}_2 . Assume, in addition, that H does not have nontrivial cyclic quotients of order dividing N (e.g., H is perfect).*

Then V is very simple if and only if the following conditions hold:

- (i) *The H -module V is absolutely simple;*
- (ii) *There do not exist a subgroup $H' \neq H$ of H and a $\mathbf{F}_2[H']$ -module V' such that V is induced by V' ;*
- (iii) *There do not exist absolutely simple $\mathbf{F}_2[H]$ -modules V_1 and V_2 , both of dimension greater than 1 and such that the H -module V is isomorphic to $V_1 \otimes_{\mathbf{F}_2} V_2$.*

Combining Theorem 7.7 with Lemma 7.4 and Theorem 7.5 we get easily the following corollary.

Corollary 7.8. *Let H be a group and V be a $\mathbf{F}_2[H]$ -module of finite dimension N over \mathbf{F}_2 . Then V is very simple if the following conditions hold:*

- (i) *The H -module V is absolutely simple;*
- (ii) *H does not contain a subgroup of finite index r with $r \mid N$ and $1 < r < N$. In addition, H does not have cyclic quotients of order N , i.e., H does not have a normal subgroup H' of index N with cyclic quotient H/H' ;*
- (iii) *There do not exist absolutely simple $\mathbf{F}_2[H]$ -modules V_1 and V_2 , both of dimension greater than 1 and such that the H -module V is isomorphic to $V_1 \otimes_{\mathbf{F}_2} V_2$.*

The following assertion follows easily from Lemma 7.4 and Theorem 7.5.

Corollary 7.9. *Suppose a positive integer $N > 1$ and a group H enjoy the following properties:*

- *H does not contain a subgroup of index dividing N except H itself.*
- *Let $N = ab$ be a factorization of N into a product of two positive integers $a > 1$ and $b > 1$. Then either there does not exist an absolutely simple $\mathbf{F}_2[H]$ -module of \mathbf{F}_2 -dimension a or there does not exist an absolutely simple $\mathbf{F}_2[H]$ -module of \mathbf{F}_2 -dimension b .*

Then each absolutely simple $\mathbf{F}_2[H]$ -module of \mathbf{F}_2 -dimension N is very simple. In other words, in dimension N the properties of absolute simplicity and very simplicity over \mathbf{F}_2 are equivalent.

The next two theorems provide examples of very simple Steinberg representations.

Theorem 7.10. *Let $q = 2^m \geq 8$ be an integral power of 2, let B be a $(q+1)$ -element set. Let G' be a group acting faithfully on B . Assume that G' contains a subgroup G isomorphic to $\mathbf{L}_2(q)$. Then the G' -module Q_B is very simple.*

Proof. We have $\mathbf{L}_2(q) = G \subset G' \subset \text{Perm}(B)$. Clearly, it suffices to check that the $\mathbf{L}_2(q)$ -module Q_B is very simple.

First, notice that $\mathbf{L}_2(q)$ acts doubly transitively on B . Indeed, each subgroup of $\mathbf{L}_2(q)$ (except $\mathbf{L}_2(q)$ itself) has index $\geq q+1 = \#(B)$ ([16], (6.27), p. 415). This implies that $\mathbf{L}_2(q)$ acts transitively on B . If the stabilizer G_b of a point $b \in B$ has index $q+1$ then it follows easily from Th. 6.25 on p. 412 of [16]) that G_b is conjugate to the (Borel) subgroup of upper-triangular matrices and therefore the $\mathbf{L}_2(q)$ -set B is isomorphic to the projective line $\mathbf{P}^1(\mathbf{F}_q)$ with the standard action of $\mathbf{L}_2(q)$ which is well-known to be doubly (and even triply) transitive. By Remark

4.5, this implies that the $\mathbf{F}_2[\mathbf{L}_2(q)]$ -module Q_B is absolutely simple. Recall that

$$\dim_{\mathbf{F}_2}(Q_B) = \#(B) - 1 = q = 2^m.$$

By Theorem 6.2, there no absolutely simple nontrivial $\mathbf{F}_2[\mathbf{L}_2(q)]$ -modules of dimension $< 2^m$. This implies that Q_B is *not* isomorphic to a tensor product of absolutely simple $\mathbf{F}_2[\mathbf{L}_2(q)]$ -modules of dimension > 1 . Recall that all subgroups in $\mathbf{L}_2(q)$ different from $\mathbf{L}_2(q)$ have index $\geq q + 1 > q = \dim_{\mathbf{F}_2}(Q_B)$. It follows from Corollary 7.8 that the G -module Q_B is very simple. Since $G \subset G'$, the G' -module Q_B is also very simple. \square

Theorem 7.11. *Let k be a positive integer and $q = 2^{2k+1}$, let B be a $(q^2 + 1)$ -element set. Let G' be a group acting faithfully on B . Assume that G' contains a subgroup G isomorphic to $\mathbf{Sz}(q)$. Then the G' -module Q_B is very simple.*

Proof. We have $\mathbf{Sz}(q) = G \subset G' \subset \text{Perm}(B)$. First, notice that $\mathbf{Sz}(q)$ acts doubly transitively on B . Indeed, the classification of subgroups of Suzuki groups ([3], Remark 3.12(e), p. 194) implies that each subgroup of $\mathbf{Sz}(q)$ (except $\mathbf{Sz}(q)$ itself) has index $\geq q^2 + 1 = \#(B)$. This implies that $\mathbf{Sz}(q)$ acts transitively on B . If the stabilizer G_b of a point $b \in B$ has index $q^2 + 1$ then it follows easily from the same classification that G_b is conjugate to the subgroup $\mathfrak{F}\mathfrak{H}$ generated by all $S(a, b)$ and $M(\lambda)$ and therefore the $\mathbf{Sz}(q)$ -set B is isomorphic to an ovoid $\mathcal{O} = \mathbf{Sz}(q)/\mathfrak{F}\mathfrak{H}$ where the action of $\mathbf{Sz}(q)$ is known to be doubly transitive ([3], Th. 3.3 on pp. 184–185 and steps g) and i) of its proof on p. 187). By Remark 4.5, this implies that the $\mathbf{F}_2[\mathbf{Sz}(q)]$ -module Q_B is absolutely simple. Recall that $\dim_{\mathbf{F}_2}(Q_B) = \#(B) - 1 = q^2 = 2^{2(2k+1)}$. By Theorem 6.3, there no absolutely simple nontrivial $\mathbf{F}_2[\mathbf{Sz}(q)]$ -modules of dimension $< 2^{2(2k+1)}$. This implies that Q_B is *not* isomorphic to a tensor product of absolutely simple $\mathbf{F}_2[\mathbf{Sz}(q)]$ -modules of dimension > 1 . Recall that all subgroups in $G = \mathbf{Sz}(q)$ (except $\mathbf{Sz}(q)$ itself) have index $\geq q^2 + 1 > q^2 = \dim_{\mathbf{F}_2}(Q_B)$. It follows from Corollary 7.8 that the G -module Q_B is very simple. Since $G \subset G'$, the G' -module Q_B is also very simple. \square

Proof of Theorem 5.4. The cases (i) and (ii) of Theorem 5.4 follow from Theorems 7.10 and 7.11 respectively applied to $G' = H$. \square

In light of Corollary 5.3 it would be interesting to classify all permutation subgroups $G \subset \text{Perm}(B)$ with very simple G -modules Q_B . We finish the paper by examples of very simple Q_B attached to Mathieu groups M_{11} and M_{12} and to related group $\mathbf{L}_2(11) = \text{PSL}_2(11)$.

Theorem 7.12. *Let n be a positive integer, B a n -element set, $G \subset \text{Perm}(B)$ a permutation group. Assume that (n, G) enjoy one of the following properties:*

- (i) $n = 11$ and G is isomorphic either to $\mathbf{L}_2(11)$ or M_{11} ;
- (ii) $n = 12$ and either $G \cong M_{12}$ or $G \cong M_{11}$ and G acts transitively on B .

Then the G -module Q_B is very simple.

Proof. Assume that $n = 11$. Since M_{11} contains a subgroup isomorphic to $\mathbf{L}_2(11)$ ([2], p. 18), it suffices to check the case of $G = \mathbf{L}_2(11)$, in light of Remark 2.2(iii).

The group $G = \mathbf{L}_2(11)$ has two conjugacy classes of maximal subgroups of index 11 and all other subgroups in G have index greater than 11 ([2], p. 7). Therefore all subgroups in G (except G itself) have index greater than 10 and the action of G on the 11-element set B is transitive. The permutation character (in both cases)

is (in notations of [2], p. 7) $1 + \chi_5$, i.e., $\chi = \chi_5$. The restriction of χ_5 to the set of 2-regular elements coincides with absolutely irreducible Brauer character φ_4 (in notations of [5], p. 7). In particular, the corresponding G -module Q_B is absolutely simple and has dimension 10. Since $10 = 2 \cdot 5$ and 5 is a prime, the very simplicity of the G -module Q_B follows from Th. 5.4 of [17]. This proves the case (i).

Assume that $n = 12$.

Suppose $G = M_{11}$ and the action of G on the 12-element set B is transitive. All subgroups G_b in G of index 12 are isomorphic to $L_2(11)$ ([2], p. 18). It follows from the already proven case (i) for $L_2(11)$ and Remark 4.4 that the G -module Q_B is very simple.

Suppose $G = M_{12}$. The action of G on the 12-element B is transitive, since all subgroups in G (except G itself) have index ≥ 12 . All subgroups G_b in G of index 12 are isomorphic to M_{11} ([2], p. 33). It follows from the already proven case (i) for M_{11} and Remark 4.4 that the G -module Q_B is very simple. \square

Combining Corollary 5.3 and Theorem 7.12 (with $B = \mathfrak{R}$, $G = \text{Gal}(f)$ and taking into account that the irreducibility of f means that $\text{Gal}(f)$ acts transitively on \mathfrak{R}), we obtain the following statement.

Theorem 7.13. *Let K be a field with $\text{char}(K) \neq 2$, K_a its algebraic closure, $f(x) \in K[x]$ an irreducible separable polynomial of degree $n \geq 5$. Let $\mathfrak{R} = \mathfrak{R}_f \subset K_a$ be the set of roots of f , let $K(\mathfrak{R}_f) = K(\mathfrak{R})$ be the splitting field of f and $\text{Gal}(f) := \text{Gal}(K(\mathfrak{R})/K)$ the Galois group of f , viewed as a subgroup of $\text{Perm}(\mathfrak{R})$. Let C_f be the hyperelliptic curve $y^2 = f(x)$. Let $J(C_f)$ be its jacobian, $\text{End}(J(C_f))$ the ring of K_a -endomorphisms of $J(C_f)$.*

Assume that n and $\text{Gal}(f)$ enjoy one of the following properties:

- (i) $n = 11$ and $\text{Gal}(f)$ is isomorphic either to $L_2(11)$ or to M_{11} ;
- (ii) $n = 12$ and $\text{Gal}(f)$ is isomorphic either to M_{11} or to M_{12} ;

Then either $\text{End}(J(C_f)) = \mathbf{Z}$ or $\text{char}(K) > 0$ and $J(C_f)$ is a supersingular abelian variety.

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